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Lagrange Interpolation Based at the Zeros of Orthonormal Polynomials with Freud Weights

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Let $L_n(f; x)$ be the Lagrange interpolation polynomial to f at the zeros of the orthonormal polynomial of degree n for the Freud weight W_Q with an exponent Q. We have the following. Let $W(\geqslant 0) \in L_1(\mathbb{R})$ and $0 < p < \infty$ be given. If for every continuous function f vanishing outside a finite interval

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} [\,|f(x) - L_n(f; x)|^p \, W(x)] \, dx = 0$$

holds, then we have

$$\int_{-\infty}^{\infty} [\, W_Q^{-1}(x)/(1 + |x|)]^p \, W(x) \, dx < \infty.$$

1. INTRODUCTION

Let Q be an even, continuous, and real-valued function defined on the real line $\mathbb{R} = (-\infty, \infty)$, and let $Q' \in C(\mathbb{R})$, $Q'(x) > 0$ on $(0, \infty)$, and Q'' be continuous on $(0, \infty)$. Furthermore, we assume that for certain constants $1 < A \leqslant B$,

$$A \leqslant \{(d/dx)(xQ'(x))\}/Q'(x) \leqslant B, \qquad x \in (0, \infty).$$

We call the function $Q(x)$ a Freud exponent, and then we consider what is called a Freud weight

$$W_Q^2(x) = \exp\{-Q(x)\}. \tag{1.1}$$

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We note that if $\alpha > 1$, then $W_\alpha^2(x) = \exp(-|x|^\alpha)$ is a Freud weight. The Mhaskar–Rahmanov–Saff number a_u is defined as the positive root of the equation

$$u = (2/\pi) \int_0^1 a_u t Q'(a_u t)(1-t^2)^{-(1/2)}\, dt, \qquad u > 0.$$

The number a_u plays an important role in the study of the approximation theory. Let Π_n denote the class of real polynomials of degree at most n, and let $\{p_n(x)\} = \{p_n(W_Q^2; x)\}$, $p_n \in \Pi_n$ be the sequence of orthonormal polynomials with respect to W_Q^2, that is,

$$\int_{-\infty}^{\infty} p_m(x)\, p_n(x)\, W_Q^2(x)\, dx = \delta_{mn} = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

We denote the zeros of $P_n(x)$ by x_{kn}, $k = 1, 2, ..., n$, where

$$x_{nn} < x_{n-1,n} < \cdots < x_{1n}.$$

Then, for a given $f \in C(\mathbb{R})$ the Lagrange interpolation polynomial $L_n(f)$ based at the zeros $\{x_{kn}\}$ of $P_n(x)$ is defined to be a unique polynomial in Π_{n-1} such that

$$L_n(f; x_{kn}) = f(x_{kn}), \qquad k = 1, 2, ..., n.$$

Nevai obtained the following.

NEVAI'S THEOREM ([12]). *Let $L_n(f; x)$ denote the Lagrange interpolation polynomial at the zeros of $P_n(W_2^2; x)$ for the weight $W_2^2(x) = \exp(-x^2)$. Let $W(\geqslant 0) \in L_1(\mathbb{R})$ and $0 < p < \infty$ be given. Suppose that for every continuous function f vanishing outside a finite interval,*

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} |f(x) - L_n(f; x)|^p\, W(x)\, dx = 0.$$

Then

$$\int_{-\infty}^{\infty} [\exp(x^2/2)/(1+|x|)]^p\, W(x)\, dx < \infty.$$

In this paper we extend Nevai's Theorem for the Freud weight (1.1).

THEOREM. *Let $W(\geqslant 0) \in L_1(\mathbb{R})$ and $0<p<\infty$ be given. If for every continuous function f vanishing outside a finite interval*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} [|f(x)-L_n(f;x)|^p W(x)]\,dx=0 \tag{1.2}$$

holds, then we have

$$\int_{-\infty}^{\infty} [W_Q^{-1}(x)/(1+|x|)]^p W(x)\,dx<\infty. \tag{1.3}$$

If in the theorem we consider especially the case of $W(x)=W_Q^p(x)(1+|x|)^{-\Delta p}$, $\Delta>1/p-1$, then obviously we have (1.3). In this case, that is, $W(x)=W_Q^p(x)(1+|x|)^{-\Delta p}$, Lubinsky and Matjila have obtained a complete solution as follows.

THEOREM OF LUBINSKY–MATJILA ([9]). *Let $1<p<\infty$, $\Delta\in\mathbb{R}$, $\alpha>0$, and $\hat{\alpha}=\min(1,\alpha)$. Then for*

$$\lim_{n\to\infty} \|\{f(x)-L_n(f;x)\}\, W_Q(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}=0$$

to hold for every continuous function $f\in C(\mathbb{R})$ satisfying

$$\lim_{n\to\infty} |f(x)|\, W_Q(x)(1+|x|)^{\alpha}=0,$$

if $p\leqslant 4$, it is necessary and sufficient that

$$\Delta> -\hat{\alpha}+1/p;$$

and if $p>4$ and $\alpha\neq 1$, it is necessary and sufficient that

$$a_n^{1/p-(\hat{\alpha}+\Delta)} n^{(1/6)(1-4/p)}=O(1), \qquad n\to\infty;$$

and if $p>4$ and $\alpha=1$, it is necessary and sufficient that

$$a_n^{1/p-(\hat{\alpha}+\Delta)} n^{(1/6)(1-4/p)}=O(1/\log n), \qquad n\to\infty.$$

Our theorem asserts that if for a certain $W(\geqslant 0)\in L_1(\mathbb{R})$,

$$\int_{-\infty}^{\infty} [W_Q^{-1}(x)/(1+|x|)]^p W(x)\,dx=\infty$$

holds, then for a continuous function f we see that $L_n(f)$ does not converge to f:

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}[|f(x)-L_n(f;x)|^p\,W(x)]\,dx\neq 0.$$

2. FUNDAMENTAL LEMMAS

Throughout this paper c will denote a positive constant independent of n and x, and the letter c will denote a constant which may differ at each different occurrence, even in the chain of inequalities. Let $c(a, b, ...)$ mean a constant depending on $a, b, ...$ By $f(x)\sim g(x)$ we denote $c_1\leqslant f(x)/g(x)\leqslant c_2$ for certain positive constants c_1, c_2, and for all relevant x.

LEMMA 2.1. *Let $|x|\leqslant ca_n$ for certain constants $c>0$. Then we have*

$$W_Q(x)\sim W_Q(x') \qquad \textit{for}\quad |x-x'|\leqslant \kappa a_n/n. \tag{2.1}$$

Proof. For $|x-x'|\leqslant\kappa a_n/n$ ($|x|$, $|x'|\leqslant ca_n$) we see

$$\begin{aligned}|Q(x)-Q(x')| &\leqslant |Q'(\xi)|\,\kappa a_n/n \qquad (x<\xi<x')\\ &\leqslant |Q'(ca_n)|\,\kappa a_n/n\\ &\leqslant c(\kappa)\end{aligned}$$

by [6, Lemma 5.1(c)]. Consequently, we see

$$\begin{aligned}W_Q(x)/W_Q(x'),\ W_Q(x')/W_Q(x) &\leqslant \exp\{|Q(x)-Q(x')|\}\\ &\leqslant \exp\{c(\kappa)\},\end{aligned}$$

that is, (2.1) follows. ∎

Let $\{p_n(x)\}=\{p_n(W_Q^2;x)\}$ be the orthonormal polynomials with respect to W_Q^2, and let $r_n=\gamma_{n-1}/\gamma_n$, where γ_n is the leading coefficient of $p_n(x)$, that is, $p_n(x)=\gamma_n x^n+\cdots$. The following lemma is useful for an estimate of values of $p_n'(x)$.

LEMMA 2.2 (cf. [2, Theorem 5], [10, Theorem 3.2]). *We have an equation*

$$p_n'(x)=A_n(x)\,p_{n-1}(x)-B_n(x)\,p_n(x),$$

where

$$A_n(x) = r_n \int_{-\infty}^{\infty} p_n^2(t)\, \bar{Q}(x, t)\, W_Q^2(t)\, dt,$$

$$B_n(x) = r_n \int_{-\infty}^{\infty} p_n(t)\, p_{n-1}(t)\, \bar{Q}(x, t)\, W_Q^2(t)\, dt,$$

and

$$\bar{Q}(x, t) = \{Q'(x) - Q'(t)\}/(x - t).$$

Proof. We can write $p_n'(x)$ in the Fourier expansion in terms of the reproducing kernel $K_n(x, t)$ as

$$p_n'(x) = \int_{-\infty}^{\infty} p_n'(t)\, K_n(x, t)\, W_Q^2(t)\, dt, \tag{2.2}$$

where

$$\begin{aligned} K_n(x, t) &= \sum_{k=0}^{n-1} p_k(x)\, p_k(t) \\ &= r_n \{p_n(x)\, p_{n-1}(t) - p_n(t)\, p_{n-1}(x)\}/(x - t). \end{aligned} \tag{2.3}$$

Since $\int_{-\infty}^{\infty} p_n(t)\{(d/dt)\, K_n(x, t)\}\, W_Q^2(t)\, dt = 0$, by (2.2) and (2.3)

$$\begin{aligned} p_n'(x) &= -\int_{-\infty}^{\infty} p_n(t)\, K_n(x, t)\{(d/dt)\, W_Q^2(t)\}\, dt \\ &= \int_{-\infty}^{\infty} p_n(t)\, K_n(x, t)\, Q'(t)\, W_Q^2(t)\, dt \\ &= -r_n \int_{-\infty}^{\infty} p_n(t)\{p_n(x)\, p_{n-1}(t) - p_n(t)\, p_{n-1}(x)\} \\ &\quad \times \bar{Q}(x, t)\, W_Q^2(t)\, dt \\ &= \left\{r_n \int_{-\infty}^{\infty} p_n^2(t)\, \bar{Q}(x, t)\, W_Q^2(t)\, dt\right\} p_{n-1}(x) \\ &\quad - \left\{r_n \int_{-\infty}^{\infty} p_n(t)\, p_{n-1}(t)\, \bar{Q}(x, t)\, W_Q^2(t)\, dt\right\} p_n(x) \\ &= A_n(x)\, p_{n-1}(x) - B_n(x)\, p_n(x). \quad \blacksquare \end{aligned}$$

LEMMA 2.3. *Let* $|x| \leqslant \delta a_n$ $(0 < \delta < 1)$. *Then for a certain constant* $c > 0$,

$$|p_n'(x)|\, W_Q(x) \leqslant cna_n^{-3/2}$$

holds.

Proof. Since by [6, (12.21)] and [6, Theorem 12.3(b)] we have

$$A_n(x) \sim n/r_n \sim n/a_n \qquad (|x| \leqslant 2a_n),$$

we see that by Schwarz's inequality

$$|B_n(x)| \leqslant \left[r_n \int_{-\infty}^{\infty} p_n^2(t)\, \bar{Q}(x, t)\, W_Q^2(t)\, dt \right]^{1/2}$$
$$\times \left[r_n \int_{-\infty}^{\infty} p_{n-1}^2(t)\, \bar{Q}(x, t)\, W_Q^2(t)\, dt \right]^{1/2}$$
$$\leqslant cn/a_n.$$

On the other hand, by [6, Corollary 1.4],

$$|p_n(x)|\, W_Q(x),\ |p_{n-1}(x)|\, W_Q(x) \leqslant ca_n^{-1/2} \qquad (|x| \leqslant \delta a_n).$$

Thus by Lemma 2.2 if $|x| \leqslant \delta a_n$, then

$$|p_n'(x)|\, W_Q(x)$$
$$\leqslant |A_n(x)|\, |p_{n-1}(x)|\, W_Q(x) + |B_n(x)|\, |p_n(x)|\, W_Q(x)$$
$$\leqslant cna_n^{-3/2}. \quad \blacksquare$$

LEMMA 2.4. *Let* $|x_{jn}|$, $|x_{j-1,n}| \leqslant \delta a_n$ $(0 < \delta < 1)$: *and let*

$$|p_n(\bar{x}_{jn})| = \max_{x_{jn} \leqslant x \leqslant x_{j-1,n}} |p_n(x)|,\ x_{jn} < \bar{x}_{jn} < x_{j-1,n}. \tag{2.4}$$

Then we have

$$\text{(i)} \quad |p_n(\bar{x}_{jn})|\, W_Q(\bar{x}_{jn}) \sim a_n^{-1/2},$$

and

$$\text{(ii)} \quad |\bar{x}_{jn} - x_{jn}|,\ |\bar{x}_{jn} - x_{j-1,n}| \sim a_n/n, \tag{2.5}$$

that is,

$$\text{(iii)} \quad x_{j-1,n} - x_{jn} \sim a_n/n.$$

Proof. (i) By [6, Corollary 1.3], for $x_{jn} < x_{j,n+1} < x_{j-1,n}$, where $|x_{jn}|$, $|x_{j-1,n}| \leqslant \delta a_n$, we see

$$|p_n(x_{j,n+1})|\ W_Q(x_{j,n+1}) \sim a_n^{-1/2}. \tag{2.6}$$

On the other hand, [6, Corollary 1.4] means

$$|p_n(x)|\ W_Q(x) \leqslant c a_n^{-1/2} \qquad (|x| \leqslant \delta a_n).$$

Therefore we have (i).

(ii) From (i) we see

$$\begin{aligned} c a_n^{-1/2}/(\bar{x}_{jn} - x_{jn}) &\leqslant |p_n(\bar{x}_{jn})|\ W_Q(\bar{x}_{nj})/(\bar{x}_{jn} - x_{jn}) \\ &= |p_n'(\xi)|\ W_Q(\bar{x}_{jn}) \qquad (x_{jn} < \xi < \bar{x}_{jn}). \end{aligned} \tag{2.7}$$

Since by [6, Corollary 1.2 (b)] we see

$$|\bar{x}_{jn} - \xi| \leqslant \kappa a_n/n$$

for some $\kappa > 0$, Lemma 2.1 means $W_Q(\bar{x}_{jn}) \leqslant c W_Q(\xi)$. Thus from (2.7) and Lemma 2.3 we have

$$c a_n^{-1/2}/(\bar{x}_{jn} - x_{jn}) \leqslant c\ |p_n'(\xi)|\ W_Q(\xi) \leqslant c n a_n^{-3/2},$$

that is,

$$c a_n/n \leqslant |\bar{x}_{jn} - x_{jn}|.$$

Consequently, we obtain $|\bar{x}_{jn} - x_{jn}| \sim a_n/n$, and similarly $|\bar{x}_{jn} - x_{j-1,n}| \sim a_n/n$.

(iii) It is trivial from (2.5). ▮

The following lemma is important itself. It gives certain exact values of $p_n(x)$ in each interval

$$I_{jn}(\delta, \varepsilon) = [x_{jn} + \varepsilon a_n/n, x_{j-1,n} - \varepsilon a_n] \cap [-\delta a_n, \delta a_n]. \tag{2.8}$$

LEMMA 2.5. *Let $I_{jn}(\delta, \varepsilon)$ be defined in (2.8). For each $0 < \delta < 1$ there exists $\varepsilon > 0$ such that*

$$x_{jn} + \varepsilon a_n/n < x_{j-1,n} - \varepsilon a_n/n, \tag{2.9}$$

whenever $|x_{jn}|$, $|x_{j-1,n}| \leqslant \delta a_n$. Then

$$|p_n(x)|\ W_Q(x) \sim a_n^{-1/2}, \qquad x \in I_{jn}(\delta, \varepsilon), \tag{2.10}$$

holds uniformly with respect to all $I_{jn}(\delta, \varepsilon) \neq \phi$.

Remark. We can also give certain exact values of $p_n'(x)$ in each interval

$$\bar{I}_{jn}(\delta, \varepsilon) = [x_{jn} - \varepsilon a_n/n, x_{jn} + \varepsilon a_n/n] \cap [-\delta a_n, \delta a_n].$$

For each $0 < \delta < 1$ there exists $\varepsilon > 0$ such that

$$|p_n'(x)| \, W_Q(x) \sim n a_n^{-3/2}, \qquad x \in \bar{I}_{jn}(\delta, \varepsilon),$$

holds uniformly with respect to all $\bar{I}_{jn}(\delta, \varepsilon) \neq \phi$.

Proof of Lemma 2.5. (i) It follows from Lemma 2.4(iii) that there exists $\varepsilon > 0$ such that (2.9) holds for every j satisfying $|x_{jn}|, |x_{j-1,n}| \leqslant \delta a_n$. Let $I_{jn}(\delta, \varepsilon) \neq \phi$, then $\bar{x}_{jn} \in [x_{jn}, x_{j-1,n}]$ is defined in (2.4). In each interval $(x_{jn}, \bar{x}_{jn})$ or $(\bar{x}_{jn}, x_{j-1,n})$ the polynomial $p_n(x)$ has at most one inflection point. We shall first consider $p_n(x)$ in $[x_{jn}, \bar{x}_{jn}]$. Then we have one of two following cases.

(a) $|p_n(x)|$ is concave on $[x_{jn}, \bar{x}_{jn}]$.

(b) There exists $x_{jn} < x_{jn}' < \bar{x}_{jn}$ such that $|p_n(x)|$ is convex on $[x_{jn}, x_{jn}']$, and concave on $[x_{jn}', \bar{x}_{jn}]$.

Let us now define the line

$$y = \{|p_n(\bar{x}_{jn})|/(\bar{x}_{jn} - x_{jn})\}(x - x_{jn}).$$

Then for the case of (a) we see

$$\{|p_n(\bar{x}_{jn})|/(\bar{x}_{jn} - x_{jn})\}(x - x_{jn}) \leqslant |p_n(x)|, \qquad x \in [x_{jn}, \bar{x}_{jn}]. \tag{2.11}$$

We shall treat the case of (b). If (2.11) is not correct, then we consider the line

$$y = |p_n'(x_{jn})| \, (x - x_{jn}),$$

and so we see

$$|p_n'(x_{jn})| \, (x - x_{jn}) \leqslant |p_n(x)|, \qquad x \in [x_{jn}, \bar{x}_{jn}]. \tag{2.12}$$

Using Lemma 2.4(i), (ii) and Lemma 2.1, the inequality (2.11) means

$$c n a_n^{-3/2}(x - x_{jn}) \leqslant |p_n(x)| \, W_Q(x), \qquad x \in [x_{jn}, \bar{x}_{jn}]. \tag{2.13}$$

If (2.12) holds, then by Lemma 2.1 and [6, Corollary 1.3] we also obtain (2.13). Hence if $x_{jn} + \varepsilon a_n/n \leqslant x \leqslant \bar{x}_{jn}$, then for a constant $c(\varepsilon)$

$$0 < c(\varepsilon) \, a_n^{-1/2} \leqslant |p_n(x)| \, W_Q(x). \tag{2.14}$$

On the other hand, from [6, Corollary 1.4]

$$|p_n(x)|\, W_Q(x) \leqslant c a_n^{-1/2}, \qquad |x| \leqslant \delta a_n,$$

that is, with (2.14) we have for $x \in [x_{jn} + \varepsilon a_n/n, \bar{x}_{jn}]$

$$|p_n(x)|\, W_Q(x) \sim a_n^{-1/2}. \tag{2.15}$$

Similarly, for $x \in [\bar{x}_{jn}, x_{j-1,n} - \varepsilon a_n/n]$ we also have (2.15). Therefore, we obtain (2.10). ∎

3. PROOF OF THEOREM

Now, we shall prove the theorem. The proof is along the same lines as Nevai's.

Proof of Theorem. We consider the space $C_0(-2, -1)$ which consists of continuous functions on $\mathbb{R}$ with support in $[-2, -1]$. By our assumption, (1.2) is satisfied for $f \in C_0(-2, -1)$. Hence for the linear functional $L_n(f)$ on $C_0(-2, -1)$ we can apply the uniform boundedness theorem (cf. Theorem 10.19 of [11, p. 182]), and so for $f \in C_0(-2, -1)$ we have

$$\int_{-\infty}^{\infty} |L_n(f; x)|^p\, W(x)\, dx \leqslant c \max_{-2 \leqslant x \leqslant -1} |f(x)|^p. \tag{3.1}$$

Let $\{p_n(x)\}$ be the orthonormal polynomials with respect to the weight $W_Q^2(x)$, and for each $n = 1, 2, 3, ...$ let us consider a function $g_n \in C_0(-2, -1)$ such that

$$\max_{-2 \leqslant x \leqslant -1} |g_n(x)| = 1, \quad g_n(x_{kn}) = \operatorname{sign} p_n'(x_{kn}), \qquad x_{kn} \in (-2, -1).$$

Then we see

$$L_n(g_n; x) = p_n(x) \sum_{-2 \leqslant x_{kn} \leqslant -1} |p_n'(x_{kn})|^{-1} (x - x_{kn})^{-1}.$$

By [6, Corollary 1.3]

$$|p_n'(x_{kn})|^{-1} \sim n^{-1} a_n^{3/2}, \qquad -2 \leqslant x_{kn} \leqslant -1,$$

and Lemma 2.4(iii)

$$\text{Num.}\, [\{k; x_{kn} \in [-2, -1]\}] \sim n/a_n,$$

where Num. $[S]$ denotes the number of elements of the set S. Thus for $x>0$

$$|L_n(g_n;x)| \geqslant c(1+x)^{-1}\,|p_n(x)|\, n^{-1}a_n^{3/2} n a_n^{-1}$$
$$= c a_n^{1/2}(1+x)^{-1}\,|p_n(x)|. \tag{3.2}$$

From (3.1) and (3.2),

$$\begin{aligned} A &= \limsup_{n\to\infty} \int_0^\infty |a_n^{1/2} p_n(x)/(1+x)|^p\, W(x)\,dx \\ &\leqslant c \limsup_{n\to\infty} \int_0^\infty |L_n(g_n;x)|^p\, W(x)\,dx \\ &\leqslant c \max_{-2\leqslant x\leqslant -1} |g_n(x)|^p \\ &= c<\infty. \end{aligned} \tag{3.3}$$

Let us define

$$I_{jn} = I_{jn}(\varepsilon) = [x_{jn} + \varepsilon a_n/n,\ x_{j-1,n} - \varepsilon a_n/n],$$
$$\bar{I}_{jn} = \bar{I}_{jn}(\varepsilon) = [x_{j-1,n} - \varepsilon a_n/n,\ x_{j-1,n} + \varepsilon a_n/n], \qquad j=2,3,4,....$$

Let $0<\delta<1$, then, from (2.14),

$$|p_n(x)|\ W_Q(x) \geqslant c a_n^{-1/2}, \qquad x \in I_{jn} \cap [0, \delta a_n],$$

where c is independent of n. Thus by (3.3)

$$c \sum_{j=2}^{n} \int_{I_{jn}\cap[0,\delta a_n]} |W_Q^{-1}(x)/(1+x)|^p\, W(x)\,dx \leqslant A. \tag{3.4}$$

Therefore, we also see that by exchanging n for $n+1$ in (3.4) we have

$$c \sum_{j=2}^{n+1} \int_{I_{j,n+1}\cap[0,\delta a_{n+1}]} |W_Q^{-1}(x)/(1+x)|^p\, W(x)\,dx \leqslant A. \tag{3.5}$$

Here, we can show that for a certain $\varepsilon>0$,

$$\bar{I}_{jn}(\varepsilon) \cap [0, \delta a_n] \subset I_{j,n+1}(\varepsilon) \cap [0, \delta a_{n+1}]. \tag{3.6}$$

In fact, by (2.6) for $|x_{j,n+1}| \leqslant \delta a_n$ there exists $c>0$ such that

$$\begin{aligned} ca_n^{-1/2} &\leqslant |p_n(x_{j,n+1})\, W_Q(x_{j,n+1})| \\ &= |\{p_n(x_{j,n+1})/(x_{j,n+1}-x_{jn})\}\, W_Q(x_{j,n+1})|\,(x_{j,n+1}-x_{jn}) \\ &\leqslant c\,|p_n'(\xi)\, W_Q(\xi)|\,(x_{j,n+1}-x_{jn}) \\ &\leqslant cna_n^{-3/2}(x_{j,n+1}-x_{jn}) \end{aligned}$$

(see Lemma 2.3). Thus we see

$$ca_n/n \leqslant (x_{j,n+1}-x_{jn}),$$

consequently, we have

$$x_{j,n+1}-x_{jn},\ x_{j-1,n}-x_{j,n+1} \sim a_n/n.$$

This means (3.6). Hence by (3.5) and (3.6)

$$\begin{aligned} &\sum_{j=2}^{n} \int_{\bar{I}_{jn}\cap[0,\delta a_n]} |W_Q^{-1}(x)/(1+x)|^p\, W(x)\, dx \\ &\quad\leqslant c \sum_{j=2}^{n+1} \int_{I_{j,n+1}\cap[0,\delta a_{n+1}]} |W_Q^{-1}(x)/(1+x)|^p\, W(x)\, dx \leqslant A. \qquad (3.7) \end{aligned}$$

Using (3.4) and (3.7), we conclude

$$\int_0^{\delta a_n} |W_Q^{-1}(x)/(1+x)|^p\, W(x)\, dx \leqslant cA,$$

where c is independent of n. Therefore

$$\int_0^{\infty} |W_Q^{-1}(x)/(1+x)|^p\, W(x)\, dx \leqslant cA. \qquad (3.8)$$

Similarly, we have

$$\int_{-\infty}^{0} |W_Q^{-1}(x)/(1+|x|)|^p\, W(x)\, dx \leqslant cA. \qquad (3.9)$$

Consequently, by (3.8) and (3.9) we have (1.3), that is, the theorem follows. ∎

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